M.Sc - Sem II PAPER

Complex Analysis

Analytic at a Plane

A single valued function $f(z)$ in a domain D is said to be analytic at a point $z = a$ if there exists a neighbourhood $|z-a| < \delta$ at all points of which the function is differentiable i.e. $f'(z)$ exists.

Analytic in a Domain, Singularities

If the above function $f(z)$ is differentiable at every point of a domain D except possibly at a finite number of exceptional points then the function is said to be analytic in the domain D. These exceptional points at which $f'(z)$ does not exist are called singular points or singularities of the function

Regular function If a function $f(z)$ be such that $f'(z)$ exists at every point of the domain D then $f(z)$ is said to be regular in D.

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Cauchy Rieman Partial differential Equation.

Th1 Necessary condition for f(z) to be analytic.

The necessary condition for $w = f(z) = u(x,y) + iv(x,y)$ to be analytic (i.e. differentiable) at any point $z_0 = x + iy$ of its domain D is that the four partial derivatives u_x, u_y, v_x and v_y should exist and satisfy the Cauchy Riemann partial differentiable equations

$$u_x = v_y \text{ and } u_y = -v_x$$

i.e. $$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Proof: Let $f(z) = u(x,y) + iv(x,y)$ be analytic at any point z of its domain therefore,

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$$f'(z) = \underset{\delta x \to 0}{Lt} \frac{f(z+\delta z) - f(z)}{\delta z}$$

exists and is unique i.e. it is independent of the path along which $\delta z \to 0$

Also $z = x + iy$ $\therefore \delta z = \delta x + i\delta y$ and ∂y also $\to 0$

$$\therefore f'(z) = \underset{\substack{\delta x \to 0 \\ \delta y \to 0}}{Lt} \frac{[u(x+\delta x, y+\delta y) + iv(x+\delta x, y+\delta y)] - [u(x,y) + iv(x,y)]}{\delta x + i\delta y}$$

$$= \underset{\substack{\delta x \to 0 \\ \delta y \to 0}}{Lt} \left[\frac{u(x+\delta x, y+\delta y) - u(x,y)}{\delta x + i\delta y} + i\frac{v(x+\delta x, y+\delta y) - v(x,y)}{\delta x + i\delta y} \right]$$

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Now let us consider two possible approaches which $\delta z \to 0$

In the first case take δz to be purely real so that $\delta z = \delta x, \delta y = 0$

and $\delta x \to 0$ Hence from (1) we get

$$f'(z) = \underset{\delta x \to 0}{Lt}\left[\frac{u(x+\delta x, y) - u(x,y)}{\delta x} + i\,\frac{v(x+\delta x, y) - v(x,y)}{\delta x}\right]$$

$$\text{or } f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = u_x + i v_x$$

Since $f'(z)$ exists therefore the above limit exists which in other words means that u_x and v_x exists

In the second case let $\delta z \to 0$ along
(I) the imaginary axis so that δz is purely imaginary and hence $\delta x = i\,\delta y$, $\delta x = 0$ and $\delta y \Rightarrow 0$

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Hence from (1) we get

$$f'(z) = \lim_{\delta y \to 0} \frac{u(x, y+\delta y) - u(x,y)}{i\delta y} + i\frac{v(x, y+\delta y) - v(x,y)}{i\delta y}$$

or $f'(z) = \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = -iu_y + v_y$ (3)

Since $f'(z)$ exists therefore the above limit exists which in other words mean that u_y and v_y exist.

Also by definition we know that the limit should be unique and hence the two limits obtained in (2) and (3) should be identical.

$$u_x + iv_x = -iu_y + v_y$$

Equating real and imaginary parts we get

$$u_x = v_y \;\&\; u_y = -v_x$$

Above equation are known as Cauchy Riemann partial differential Equation.